REPORT DOCUMENTATION PAGE

Form Approved OMB NO. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
11-01-2010	Final Report	15-Jul-2006 - 14-Jul-2008

4. TITLE AND SUBTITLE
Controlling interacting systems in noisy environments

5a. CONTRACT NUMBER
W911NF-06-1-0320

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
611102

6. AUTHORS
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5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAMES AND ADDRESSES
Montclair State University
1 Normal Ave.

Montclair, NJ 07043 -

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSOR/MONITOR'S ACRONYM(S)
ARO

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
49444-MA.1

12. DISTRIBUTION AVAILABILITY STATEMENT
Approved for Public Release; Distribution Unlimited

13. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not contrued as an official Department of the Army position, policy or decision, unless so designated by other documentation.

14. ABSTRACT
The objective is to develop concepts for the analysis of the dynamics of interacting systems in a noisy environment. One of the central issues is dynamics of noise-induced switching. The phenomenon underlies a large portion of all significant changes that occur in systems in noisy environment. Examples range from breakdown events in complex systems to swarming in systems of interacting vehicles to overcoming barriers by such vehicles. Therefore understanding the switching dynamics is instrumental for developing highly efficient ways of controlling noisy

15. SUBJECT TERMS
multi-particle dynamics, stochastic effects, optimal control

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Lora Billings
UU	UU	UU	UU		19b. TELEPHONE NUMBER
					973-655-7812

1 Foreward

Developing mathematical tools for describing the collective motion of multi-agent systems is critical for new approaches that should enhance Army operations, since such tools should ultimately lead to efficient means of controlling the collective motion. Multi-agent systems originate from various applications, from biology to physics to transportation to multi-robot systems. They display complex behavior [14, 25, 7, 37], with pattern formation and swarming as observed in biological populations including bacterial colonies [1, 3, 4], slime molds [22, 27], locusts [13] and fish [8]. Mathematical studies of this behavior have been performed for a few decades. The information gained from these studies has already led to an increased ability to intelligently design and control man-made vehicles [9, 19, 21, 26, 35]. The mathematical models can capture emergent properties of numerous existing and future applications of military and industrial platforms.

Several types of mathematical models have been used to describe coherent patterns and swarms. One popular approach is based on a continuum approximation in which scalar and vector fields are use d for the relevant quantities [13, 16, 32, 33, 34]. Another popular approach is based on treating every individual or object as a discrete particle [8, 15, 16, 33, 36]. Such many-particle systems typically have their own dynamics, but interact with others. In many programs of DoD interest, interacting particles in external static or time-dependent potentials is a central theme in the construction and analysis of organized behavior. Dynamically interacting autonomous swarmer (DIAS) systems are comprised of a multitude of simple autonomous vehicles, which are loosely coupled via communication. They will play a key role in future deployments, as the drive to miniaturize electronic devices results in smaller and more capable self-mobile machines with limited decision making abilities.

A basic physical principle is that, as systems become smaller, an increasingly important role is played by external fluctuations in the environment. Interacting particles subject to external fluctuations but coupled through communication needs to be understood from a point of view of stochastically modulated many-body dynamical systems. A major mathematical challenge comes from the fact that the regular motion, without fluctuations, has multiple attractors, and fluctuations cause inter-attractor switching, thus bringing additional time scales associated with the switching rates. The delicate interplay of interaction, fluctuations, and multi-stability provides a foundation for pattern formation. Understanding this interplay is a key to the control of the many-particle dynamics. Analysis of stochastic interacting dynamical systems would lead to streamlined computer, communication, surveillance, and reconnaissance systems.

This research couples the rapidly developing areas of nonlinear phenomena, fluctuations, and dissipative systems far from thermal equilibrium. A key concept in the analysis of motion is noise-activated escape from a metastable state of local equilibrium, e.g. a state at the minimum of a potential well. Such escape leads to an interstate switching. Activated processes underlie many fundamental phenomena in nature, such as diffusion in solids, nucleation, and protein folding. Much less is known about switching of systems away from thermal equilibrium, especially those driven by time-dependent potentials used in swarming and pattern-forming models. It is poorly understood whether escape rates of nonequilibrium systems should display any universal scaling dependence on control parameters at all. Gaining an insight into escape in driven systems requires that one knows how a system *moves* in an activated process. Even though an activated process happens at random, when it does happen the system is most likely to follow a certain dynamical path [18, 10, 24, 28, 5]. Knowing this path opens the way for control over activated processes. Such knowledge is useful for random clustering of dynamical patterns. Much of the motivation comes from our recent evidence that activated processes that induce escape can be selectively controlled.

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2 Statement of the problem studied

The objective of this project is to develop concepts for the analysis of the dynamics of interacting systems in a noisy environment. New approaches should lead to a better understanding of system dynamics and generate novel efficient algorithms of stochastic optimal control for interacting systems.

One of the central issues that we address is dynamics of noise-induced switching. The phenomenon underlies a large portion of all significant changes that occur in systems in noisy environment. Examples range from breakdown events in complex systems to swarming in systems of interacting vehicles to overcoming barriers by such vehicles. Therefore understanding the switching dynamics is instrumental for developing highly efficient ways of controlling noisy systems.

Central to the theoretical approach is the notion that the dynamical trajectories followed in switching form narrow tubes. We demonstrate that the tubes can be directly observed in experiment. Quantitatively, the tubes are characterized by the distribution of trajectories. To find it theoretically we modify the instanton technique developed in a completely different area, the quantum field theory. This approach maps the problem of most probable switching trajectories in noisy dissipative systems onto a problem of Hamiltonian dynamics of an auxiliary system of a higher dimension.

This award facilitated the formation of the new team of investigators consisting of an applied mathematician, a theoretical physicist, and an experimental physicist. It also included collaboration with applied mathematician Ira Schwartz at the Naval Research Laboratory. Utilizing the complementary skill sets, the group produced significant results in physics, mathematics, and the life sciences.

3 Summary of the most important results

The following is a brief summary of the most important results:

1. We have solved the long-standing problem of noise-induced switching in periodically modulated systems. We found the distribution of trajectories followed in switching. This distribution may display several peaks separated by the modulation period. Analytical results agree with the results of simulations of a Brownian particle in a model modulated potential.

2. We have solved the problem of control of distribution over period-two states. We show that a comparatively weak field can strongly affect the rates of switching between the

states. The logarithm of the rate change is linear in the control field amplitude. We predict lifting of degeneracy of period-two states and the possibility of an extremely strong fluctuation-mediated wave mixing.

3. To interpret switching events in the driven colloidal system, it is necessary to track each object in an ensemble of nominally identical objects. We have developed algorithms to identify large fluctuations in the space of particle coordinates. The method forms a predictor conditioned on the existing flow and Gaussian noise amplitude, then flags events with low probability. The algorithm searches for correlations among the particles that precede switching events. The approach can be extended to include non-Gaussian noise and periodic modulation of the optical field.

4. We have made significant progress in developing a general mathematical approach to the analysis of switching in systems driven by a Gaussian and a non-Gaussian noise. Our preliminary results indicate the possibility of exponentially strong effect of a non-Gaussian noise on the switching rate. The effect can be expressed in a closed form in terms of the characteristic functional of the noise, which is important for many applications. We have started studying specific important types of non-Gaussian noise, and in particular, shot noise. Analytical results agree with the results of simulations of a Brownian particle in a model modulated potential.

5. We have developed a formulation that allows one to observe switching trajectories and find the most probable paths without making any preliminary assumptions about the system. This formulation has been tested experimentally using a mesoscopic system of significant interest, a micro-electro-mechanical oscillator. All major theoretical predictions have been fully confirmed in the experiment, see Fig. 1.

6. We have developed a theory of disease extinction in large populations and discovered a new scaling behavior of the extinction rate with the varying reproduction rate of infection. We have also developed a mathematical approach that allowed us to predict and describe the exponentially strong effect of random vaccination on disease extinction.

7. We have developed a theory and analysis of delayed communication in stochastic swarms to examine the effect of latency in coordinated behavior of multiple vehicles. We show that with the addition of a time delay, the model possesses a transition that depends on the size of the coupling amplitude. This transition is independent of the initial swarm state (traveling or rotating) and is characterized by the alignment of all of the individuals along with a swarm oscillation. See Fig. 2.

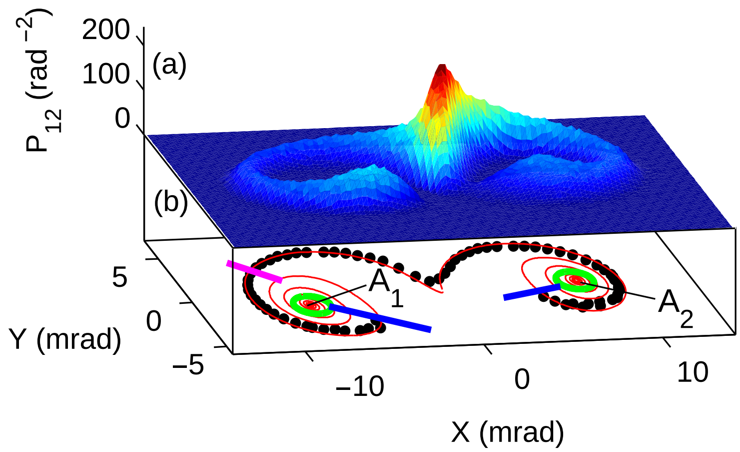


Figure 1: (a) Switching probability distribution in a parametrically driven micro-electro-mechanical oscillator. The probability distribution $P_{12}(X, Y)$ is measured for switching out of state A_1 into state A_2. (b) The peak locations of the distribution are plotted as black circles and the theoretical most probable switching path is indicated by the red line. All trajectories originate from within the green circle in the vicinity of A_1 and later arrive at the green circle around A_2. The portion of the distribution outside the blue lines is omitted.

8. We show that steady-state work fluctuations in periodically modulated systems display universal features, which are not described by the standard fluctuation theorems. Modulated systems often have coexisting stable periodic states. We find that work fluctuations sharply increase near a kinetic phase transition where the state populations are close to each other. This exponential peak is a new strongly pronounced phenomenon which has not been previously appreciated. We also show that the work variance displays scaling with the distance to a bifurcation point where a stable state disappears and find the critical exponent for a saddle-node bifurcation. See Fig. 3.

9. We explore the distribution of paths followed in fluctuation-induced switching between coexisting stable states. We introduce a quantitative characteristic of the path distribution in phase space that does not require a priori knowledge of system dynamics. The theory of the distribution is developed and its direct measurement is performed in a micromechanical oscillator driven into parametric resonance. The experimental and theoretical results on the shape and position of the distribution are in excellent agreement, with no adjustable parameters. In addition, the experiment provides the first demonstration of the lack of time-reversal symmetry in switching of systems far from thermal equilibrium. The results open the possibility of efficient control of the

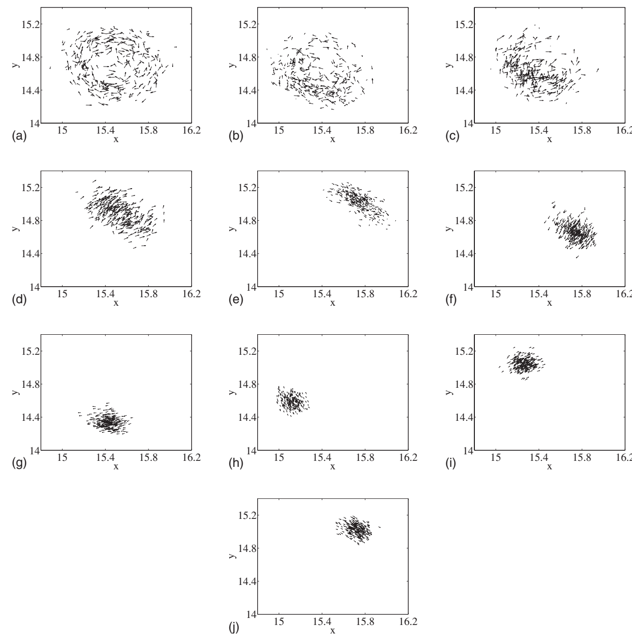
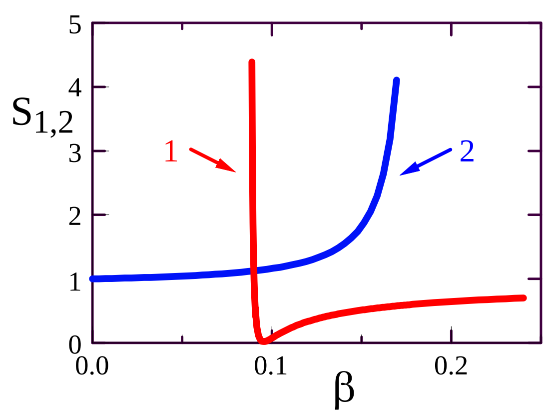


Figure 2: Snapshots of a swarm taken at (a) t =50, (b) t =60, (c) t =62, (d) t =64, (e) t =66, (f) t =68, (g) t =70, (h) t =72, (i) t =74, and (j) t =76, with N=300 particles, and noise intensity, D=0.08. The swarm was in a rotational state when the time delay of 1 was switched on at t =40.



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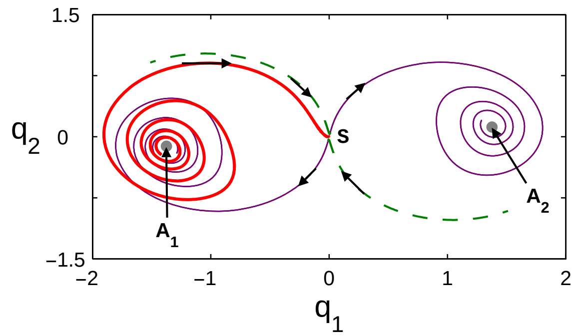

Figure 4: Phase portrait of a two-variable system with two stable states A_1 and A_2. The saddle point $\mathcal{S}$ lies on the separatrix that separates the corresponding basins of attraction. The thin solid lines show the downhill deterministic trajectories from the saddle to the attractors. A portion of the separatrix near the saddle point is shown as the dashed line. The thick solid line shows the most probable trajectory that the system follows in a fluctuation from A_1 to the saddle. The most probable switching path (MPSP) from A_1 to A_2 is comprised by this uphill trajectory and the downhill trajectory from $\mathcal{S}$ to A_2. The plot refers to the system studied experimentally, an underdamped nonlinear parametrically modulated oscillator with the modulation frequency close to twice the eigenfrequency.

switching probability based on the measured narrow path distribution. See Fig. 4.

10. Population extinction is of central interest for population dynamics. It may occur from a large rare fluctuation. We find that, in contrast to related large-fluctuation effects like noise-induced interstate switching, quite generally extinction rates in multipopulation systems display fragility, where the height of the effective barrier to be overcome in the fluctuation depends on the system parameters nonanalytically. We show that one of the best-known models of epidemiology, the susceptible-infectious-susceptible (SIS) model, is fragile to total population fluctuations. See Figs. 5 and 6.

11. We have investigated the stochastic extinction processes in a class of epidemic models and showed that the effective entropic barrier for extinction in a susceptibleinfected-susceptible epidemic model displays scaling with the distance to the bifurcation point, with an unusual critical exponent. We make a direct comparison between predictions and numerical simulations. We also consider the effect of non-Gaussian vaccine schedules, and show numerically how the extinction process may be enhanced when the vaccine schedules are Poisson distributed. See Fig. 7.

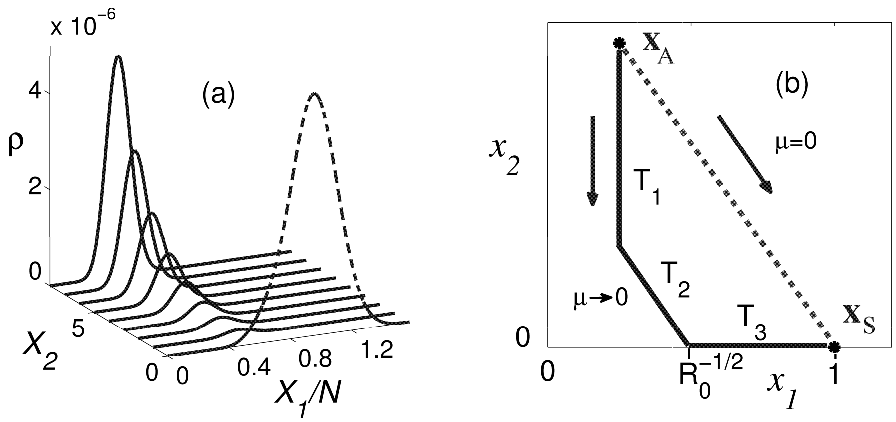

Figure 5: A snapshot of the probability $\rho(\mathbf{X})$ near the extinction plane $X_2 = 0$ for the SIS model; ρ is quasi-continuous in X_1/N, with X_1 and X_2 being the total numbers of susceptibles and infected, respectively, and N being the charateristic total population. The data of simulations refer to $\mu t = 9, R_0 = 4, \mu' \equiv \mu/(\mu + \kappa) = 0.1$, where μ is the birth-death rate and R_0 is the infection reproduction rate. For $t = 0$ the system was at the stabl;e state $\mathbf{X}_A$, the total number of particles was $N = 50$. (b) Asymptotic optimal trajectories for extinction for $\mu \to 0$ (solid line) and $\mu = 0$ (dashed line).

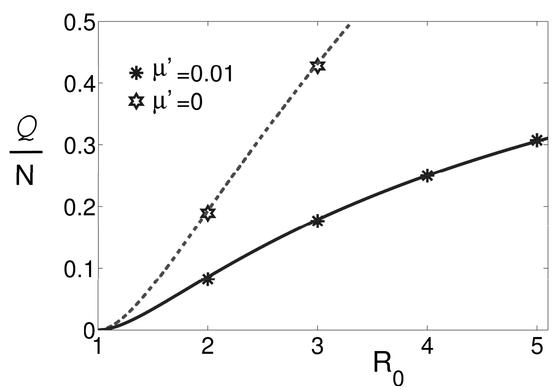

Figure 6: The switching exponent $\mathcal{Q}$ for the SIS model of epidemics. The epidemics extinction rate is $W \propto \exp(-\mathcal{Q})$. The solid and dashed lines show the results for the birth-death rate $\mu \to 0$ and $\mu = 0$, respectively. The data points are obtained from the numerical solution of the master equation for the total initial populations $N = 50$ and $N = 100$, which made it possible to directly extract the exponent $\mathcal{Q}$.

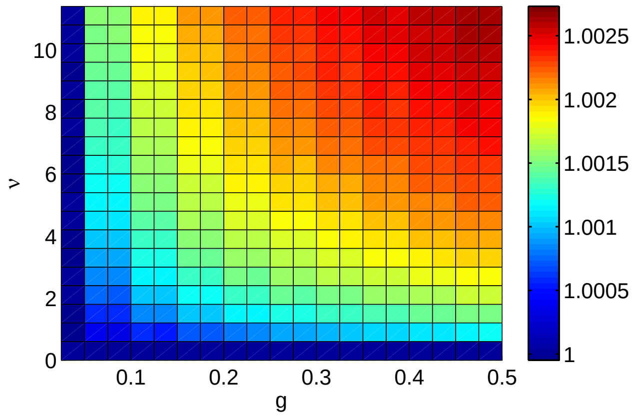

Figure 7: Extinction factor increase as a function of vaccination parameters numerically computed using the Monte Carlo simulation. Parameters shown are g =vaccination amplitude (percentage of susceptibles) and ν =vaccination frequency per year. The population size is $N = 1000$.

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